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Gap Between Multiobjective and Scalar Objective Optimization Problem: Second Order Case

H. S. Faruque Alam[1]
[1]Department of Mathematics, University of Chittagong, Bangladesh

Email address:
hsfaruque_math_cu@yahoo.com

Abstract: We establish the reasons for the existence of multi-objective and scalar objective optimization for the second order case with a suitable example. We know that Guignard's constraint qualification (GCQ) are the weakest assumption guaranteeing that the necessary conditions of the Karush-Kuhn-Tucker (KKT) type hold in single objective optimization problem nonetheless it is also true for second order single objective optimization. However, GCQ is not used for both first and second order multi-objective cases, and in this paper, we present the appropriate reasons behind it with an example.

Keywords: Multi-objective optimization, Scalar objective optimization, Guignard's constraint qualification (GCQ), Karush-Kuhn-Tucker (KKT).

1. Introduction

Investigation of multi-objective optimality conditions has been one of the most recent topics in the theory of optimization problems. Many authors gave first order optimality conditions of vector objective optimization by using several constraint qualifications [5, 6] and also presented the gap between single and multi-objective optimization for the first order case [2, 9]. Unpredictably, not many papers have been devoted to study second order optimality criteria. According to [4], the first paper concerning second order necessary conditions for multi-objective problems with a set constraint appeared in [3] and then [7]. Relying on [1,3,8] we have demonstrated the reasons for the existence of multi-objective and scalar objective optimization for the second order case by a suitable example.

2. Preliminaries

2.1 Basic notions

We consider the following multi objective optimization problem P_1:

$$\min \mathbf{f}(\mathbf{x}), \text{ subject to the set X:}$$

$$\bar{\mathbf{x}} \in \mathrm{X} = \left\{ \mathbf{x} \in \mathrm{E}_n \middle| \mathbf{g}(\mathbf{x}) \leqq 0, \mathbf{h}(\mathbf{x}) = 0 \right\}$$

Let, $f : \mathrm{E}_n \to \mathrm{E}_1$, $g : \mathrm{E}_n \to \mathrm{E}_m$ and $h : \mathrm{E}_n \to \mathrm{E}_k$ *be* twice continuously differentiable vector-valued functions. Assume that $I(\bar{\mathbf{x}}) = \{ j : g_j(\bar{\mathbf{x}}) = 0 \}$ *for j=1,…,m.*

For any twice continuously differentiable function $\mathbf{g} : \mathrm{E}_n \to \mathrm{E}_m$ and for any vector $\mathbf{d} \in \mathrm{E}_m$, we denote by $\nabla \mathbf{g}(\bar{\mathbf{x}})$ and $\nabla^2 \mathbf{g}(\bar{\mathbf{x}})(\mathbf{d}, \mathbf{d})$ respectively the $m \times n$ Jacobian matrix and the m-dimensional vector whose *i*th component is $\mathbf{d}^T \nabla^2 g_i(\bar{\mathbf{x}}) \mathbf{d}$.

In this paper we use the following notions:

For any two vectors $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{y} = (y_1, y_2)^T$ in E_2, we use the following conventions:

$\mathbf{x} \leqq_{lex} \mathbf{y}$ means that $x_1 < y_1$ holds or $x_1 = y_1$ and $x_2 \leqq y_2$ hold.

$\mathbf{x} <_{lex} \mathbf{y}$ means that $x_1 < y_1$ holds or $x_1 = y_1$ and $x_2 < y_2$ hold.

The subscript *lex* means lexicographic order.

The following well-known second order approximation of a set provides the satisfactory tool, which is the modification of the first order contingent cone. For detail [4].

Definition 2.1 The second order contingent set to X at $\bar{\mathbf{x}} \in \mathrm{clX}$ in the direction $\mathbf{d} \in \mathrm{E}_n$ is the set defined by $(\mathbf{d}, \mathbf{z}) \in \mathrm{cl\,conv} \mathrm{T}^2(\mathrm{X}; \bar{\mathrm{x}}; \mathrm{d})$ (Second order **Guignard's**

$$\mathrm{T}^2(\mathrm{X};\bar{\mathrm{x}};\mathrm{d}) \equiv \left\{(\mathbf{d},\mathbf{z}) \in \mathrm{E}_n \middle| \exists \mathbf{x_n} \in \mathrm{X}, \exists t_n \to +0 \text{ such that } \mathbf{x_n} = \bar{\mathbf{x}} + t_n\mathbf{d} + \frac{1}{2}t_n^2\mathbf{z} + o(t_n^2)\right\}$$

where $o(t_n^2)$ is a vector satisfying $\frac{\left\|o(t_n^2)\right\|}{\mathrm{t}_n^2} \to 0$.

In general the second order contingent is not a cone and it does not preserve convexity and also $\mathrm{T}^2(\mathrm{X};\bar{\mathrm{x}};\mathrm{d})$ is empty whenever $\mathbf{d} \notin \mathrm{T}(\mathrm{X},\bar{\mathrm{x}})$ but the converse does not hold. [4, 8].

A first order sufficient condition for vector minimum point is that the following system has no nonzero solution **d.**

$$\nabla\mathbf{f}(\bar{\mathbf{x}})^T\mathbf{d} \leqq 0, \nabla g_I(\bar{\mathbf{x}})\mathbf{d} \leqq 0, \nabla h(\bar{\mathbf{x}})\mathbf{d} = 0 \qquad 2.1$$

The Kuhn-Tucker type condition for optimality is equivalent to the inconsistency of the following system:

$$\nabla\mathbf{f}(\bar{\mathbf{x}})^T\mathbf{d} < 0, \nabla\mathbf{g}_I(\bar{\mathbf{x}})^T\mathbf{d} \leqq 0, \nabla\mathbf{h}(\bar{\mathbf{x}})^T\mathbf{d} = 0 \qquad 2.2$$

The gap between (2.1) and (2.2) is caused by the following directions:

$$\nabla\mathbf{f}(\bar{\mathbf{x}})^T\mathbf{d} \leqq 0, \nabla f_i(\bar{\mathbf{x}})^T\mathbf{d} = 0 \text{ at least one } i,$$

$$\nabla\mathbf{g}_I(\bar{\mathbf{x}})^T\mathbf{d} \leqq 0,\ \nabla\mathbf{h}(\bar{\mathbf{x}})^T\mathbf{d} = 0 \qquad 2.3$$

A direction **d** that satisfies (1.3) is called a critical direction. For the sake of simplicity, we use the following notations:

$$F_i(\mathbf{d},\mathbf{z}) = \left(\nabla f_i(\bar{\mathbf{x}})^T\mathbf{d}, \nabla f_i(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{d},\mathbf{d})\right)^T$$

$$G_j(\mathbf{d},\mathbf{z}) = \left(\nabla g_j(\bar{\mathbf{x}})^T\mathbf{d}, \nabla \mathrm{g}_j(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 g_j(\bar{\mathbf{x}})(\mathbf{d},\mathbf{d})\right)^T$$

$$H_p(\mathbf{d},\mathbf{z}) = \left(\nabla h_p(\bar{\mathbf{x}})^T\mathbf{d}, \nabla h_p(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 h_p(\bar{\mathbf{x}})(\mathbf{d},\mathbf{d})\right)^T$$

2.2 Second order necessary conditions

Now, consider a problem P_2:

$\min \mathbf{f}(\mathbf{x})$, subject to the set $\mathbf{x} \in \mathrm{X}$.

Lemma 2.1 If $\bar{\mathbf{x}} \in \mathrm{X}$ is an efficient solution of P_2 then for any direction $\mathbf{d} \in \mathrm{T}(\mathrm{X};\bar{\mathrm{x}}) \cap \left\{\nabla f_i(\bar{\mathbf{x}})^T\mathbf{d} = 0\right\}$ the system

$$F_i(\mathbf{d},\mathbf{z}) <_{lex} 0,\ \forall i \qquad 2.4$$

has no solution $(\mathbf{d},\mathbf{z}) \in \mathrm{T}^2(\mathrm{X};\bar{\mathrm{x}};\mathrm{d})$.

i.e. $F_i(\mathbf{d},\mathbf{z}) \cap \mathrm{T}^2(\mathrm{X};\bar{\mathrm{x}};\mathrm{d}) = \phi$

Proof can be seen in [3, 4]

In the single objective case, that is, $l = 1$, the Lemma 2.1 holds at the optimal point considering any direction **constraint qualification (GCQ))**. This is no longer true in the multi objective case, as the following example shows.

Example 2.1. Consider the problem

$$\min\left\{-\mathrm{x}_2 - \frac{1}{2}\mathrm{x}_1^2, \mathrm{x}_2 - \mathrm{x}_1^2\right\} \text{and } \mathrm{X} = \left\{(\mathrm{x}_1,\mathrm{x}_2) \middle| 2\,\mathrm{x}_1^4 \leqq \mathrm{x}_2^2 \leqq 4\mathrm{x}_1^4\right\}$$

It is easily verified that:

i) $\mathrm{x}_0 = (0,0)$ is an efficient solution to the problem.

ii) Choose $\mathbf{d} = (1,0) \in \mathrm{T}(\mathrm{X},\bar{\mathrm{x}})$ where $\nabla f_i(\bar{\mathbf{x}})^T\mathbf{d} = 0\ \ \forall i$ and $\mathrm{T}^2(\mathrm{X};\bar{\mathrm{x}};\mathrm{d}) = \left\{(\mathrm{z}_1,\mathrm{z}_2) \mid 2\sqrt{2} \leqq |\mathrm{z}_2| \leqq 4\right\}$.

iii) $F_i(\mathbf{d},\mathbf{z}) <_{lex} 0$,

$$\Rightarrow \nabla f_i(\bar{\mathbf{x}})^T\mathbf{d} = 0 \text{ and } \nabla f_i(\bar{\mathbf{x}})^T\mathbf{z} + \nabla^2 f_i(\bar{\mathbf{x}})(\mathbf{d},\mathbf{d}) < 0, \forall i$$

$$\Rightarrow \nabla f_1(\bar{\mathbf{x}})^T\mathbf{z} + \mathbf{d}^{\mathbf{T}}\nabla^2 f_1(\bar{\mathbf{x}})\mathbf{d} < 0 \text{ and}$$

$$\nabla f_2(\bar{\mathbf{x}})^T\mathbf{z} + \mathbf{d}^{\mathbf{T}}\nabla^2 f_2(\bar{\mathbf{x}})\mathbf{d} < 0$$

$$\Rightarrow (0,-1)^{\mathrm{T}}\begin{pmatrix}\mathrm{z}_1\\ \mathrm{z}_2\end{pmatrix} + (1,0)^{\mathrm{T}}\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} < 0 \text{ i.e. } \mathrm{z}_2 > -1$$

And

$$(0,1)^{\mathrm{T}}\begin{pmatrix}\mathrm{z}_1\\ \mathrm{z}_2\end{pmatrix} + (1,0)^{\mathrm{T}}\begin{pmatrix}-2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} < 0, \text{ i.e. } \mathrm{z}_2 < 2$$

i.e. $F_i(\mathbf{d},\mathbf{z}) = \left\{(\mathrm{z}_1,\mathrm{z}_2) \mid -1 < \mathrm{z}_2 < 2\right\} <_{lex} 0$

iv) $F_i(\mathbf{d},\mathbf{z}) \cap \mathrm{T}^2(\mathrm{X};\bar{\mathrm{x}};\mathrm{d}) = \phi$

v) But $\mathrm{cl\ conv}\mathrm{T}^2(\mathrm{X};\bar{\mathrm{x}};\mathrm{d}) = \left\{(\mathrm{z}_1,\mathrm{z}_2) \mid |\mathrm{z}_2| \leq 4\right\}$

vi) So, $F_i(\mathbf{d},\mathbf{z}) \cap \mathrm{cl\ conv}\mathrm{T}^2(\mathrm{X};\bar{\mathrm{x}};\mathrm{d}) \neq \phi$

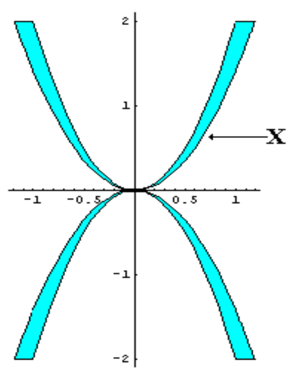

Fig. 1

REMARK 2.1 : In the above example the only challenge is to compute the contingent set; the hint is: suppose a direction w is in the contingent set and write down that the moving point (given from the direction) satisfies the constraints; take the limit in these expressions and reader get necessary condition for **w** to be in the contingent set. Finally, test the sufficiency of these conditions, just showing concrete sequences of t_n and w_n for a generic direction **w** which satisfies them.

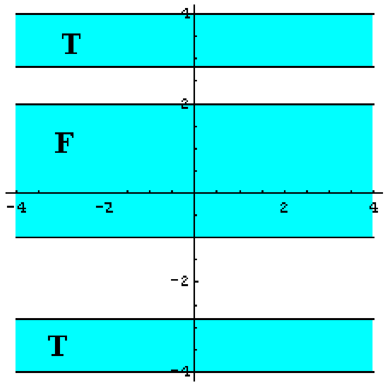

Fig. 2

3. CONCLUSION

In Lemma 2.1 we see that relation (2.1) holds for any objective function, but if we replace the second order contingent cone by $(\mathbf{d},\mathbf{z}) \in \mathrm{cl}\,\mathrm{conv}\mathrm{T}^2(\mathrm{X};\bar{\mathrm{x}};\mathrm{d})$ then the lemma does not hold. However, convexlikeness of the objective function guarantees that (2.1) is held at the considered optimal point for all the directions in the closed convex hull of the second order contingent set.

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